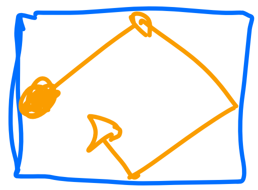
(1)

$$\boxed{\underbrace{\partial_t f_1(\vec{q}_1,\vec{p}_1,t) + \{f_1, H_1\}}_{\text{free evolution}} = \underbrace{\int d\vec{q}_1 d\vec{p}_2 \, \frac{\partial V(\vec{q}_1-\vec{q}_2)}{\partial \vec{q}_1} \cdot \frac{\partial f_2(\vec{q}_1,\vec{p}_1,\vec{q}_2,\vec{p}_2)}{\partial \vec{p}_1}}_{\text{interaction term}}} \qquad (F1)$$

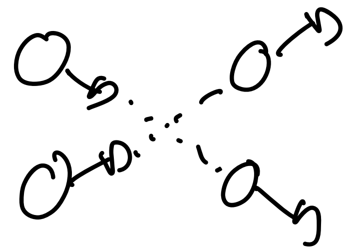
Relevant time scales:

The free evolution. $\tau_F \simeq 2.10^{-3} s$

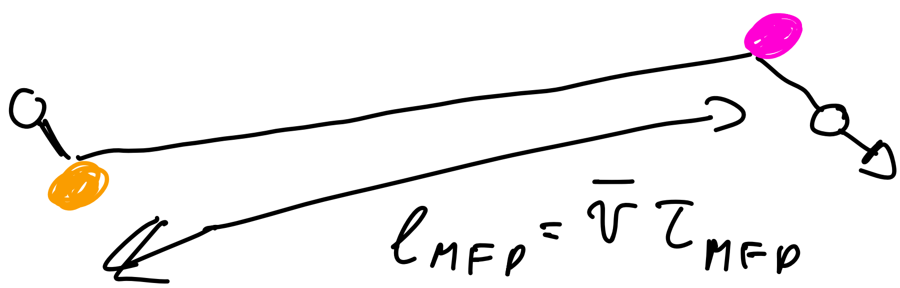


Interaction term

(1) duration of a collision $\tau_{col} \simeq 6.10^{-13} s = 0{,}6 \text{ ps}$



(2) time between collisions $\tau_{MFP} \simeq 3 \; 10^{-10} s$



$$\ell_{MFP} = \bar{v}\,\tau_{MFP}$$

Three well separated time scales

$$\tau_{col} \simeq 6.10^{-13} \ll \tau_{MFP} \simeq 3.10^{-10} s \ll \tau_F \simeq 2.10^{-3} s$$

Boltzmann equation

Describe the evolution over a time τ such that $\tau_{col} \ll \tau \ll \tau_{MFP}$

→ collisions look instantaneous

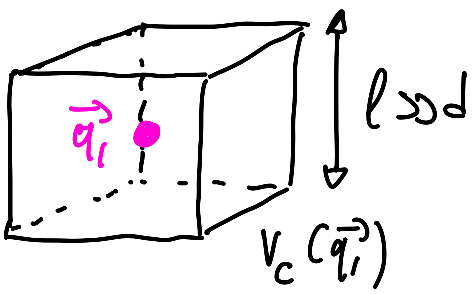
→ ——— are rare & random events, leading to small variations to f_1

2.2.2) The coarse graining

Start from (F1) & build a coarse grained description over scales τ, ℓ such that

$$\tau_{MFP} \gg \tau \gg \tau_{col} \quad \& \quad \ell_{MFP} \gg \ell \gg d$$

2.2.2.1) Spatial coarse graining



Define $\hat{f}_1(\vec{q}_1, \vec{p}_1, t)$ the average number density of particles in the box with momentum $\vec{p}_1$

$$\hat{f}_1(\vec{q}_1, \vec{p}_1, t) = \frac{1}{V_c} \int_{V_c(\vec{q}_1)} d^3\vec{q} \; f_1(\vec{q}, \vec{p}_1, t) \qquad (CG)$$

Comment: Defining $K(\vec{u})$ a top-hat filter such that

$K(\vec{u}) = 1$ if $\vec{u}$ is in a box of volume V_c centered at $\vec{0}$

$\phantom{K(\vec{u})} = 0$ otherwise

$\hat{f}_1(\vec{q}_1,\vec{p}_1,t) = K * f_1(\vec{q}_1,\vec{p}_1,t)$ when $f*g(\vec{q}) = \int d^3\vec{r}\, f(\vec{q}-\vec{r})\, g(\vec{r})$

f_1 may vary rapidly with $\vec{q}_1$, but $\hat{f}_1$ is smoothed out by the convolution. This is what one expect from a coarse grained field.

Time evolution of $\hat{f}_1(\vec{q}_1,\vec{p}_1,t)$:

$\frac{1}{V_c}\int_{V_c(q_1)} d\vec{q}\ (F_1)$

$$\frac{1}{V_c}\int_{V_c(\vec{q}_1)} d\vec{q}\ \partial_t f_1(q,\vec{p}_1,t) = \partial_t \hat{f}_1(\vec{q}_1,\vec{p}_1,t)$$

$$= \underbrace{-\frac{1}{V_c}\int d\vec{q}\ \{f_1,H_1\}}_{\text{① Free evolution } \equiv\, \partial_t \hat{f}_1|_{Free}} + \underbrace{\frac{1}{V_c}\int d\vec{q}\int d\vec{q}_2\, d\vec{p}_2\ \frac{\partial V(\vec{q}-\vec{q}_2)}{\partial\vec{q}}\cdot\frac{\partial f_2(\vec{q}_1,\vec{p}_1,\vec{q}_2,\vec{p}_2)}{\partial\vec{p}_1}}_{\text{② interaction term } \equiv\, \partial_t \hat{f}_1|_{col}}$$

A) The free evolution

Since nothing happens on scales τ, ℓ due to H_1, we expect that

$$\partial_t \hat{f}_1|_{Free} = \{\hat{f}_1, H_1\}$$

$$(*)\quad \frac{1}{V_c}\int_{V_c(\vec{q}_1)} d\vec{q}\ \underbrace{\frac{\partial H(\vec{q},\vec{p})}{\partial\vec{q}_1}}_{\simeq\ \text{constant in } V_c}\cdot\frac{\partial}{\partial\vec{p}_1} f_1 \simeq \frac{\partial H(\vec{q}_1,\vec{p}_1)}{\partial\vec{q}_1}\cdot\frac{\partial}{\partial\vec{p}_1}\frac{1}{V_c}\int d\vec{q}\, f_1 = \frac{\partial H}{\partial\vec{q}_1}\cdot\frac{\partial\hat{f}}{\partial\vec{p}_1}$$

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$$\circledast\; -\frac{1}{V_c}\int d\vec{q}\,'\,\frac{\partial H}{\partial \vec{p}_1}\cdot\frac{\partial f_1}{\partial \vec{q}_1}(\vec{q},\vec{p}_1,t) = -\frac{\partial H}{\partial \vec{p}_1}\cdot\left[\frac{1}{V_c}\int d\vec{q}\,'\,\frac{\partial f_1}{\partial \vec{q}_1}(\vec{q},\vec{p}_1,t)\right]$$

By linearity, the spatial average of the gradient is the gradient of the spatial average (*)

$$\Rightarrow \frac{1}{V_c}\int d\vec{q}\;\frac{\partial f_1}{\partial \vec{q}_1}(\vec{q},\vec{p}_1,t) = \frac{\partial}{\partial \vec{q}_1}\left[\frac{1}{V_c}\int d\vec{q}\; f_1(\vec{q},\vec{p},t)\right]$$

$$\Rightarrow -\frac{1}{V_c}\int d\vec{q}\,'\,\frac{\partial H}{\partial \vec{p}_1}\cdot\frac{\partial f_1}{\partial \vec{q}_1}(\vec{q},\vec{p}_1,t) = -\frac{\partial H}{\partial \vec{p}_1}\cdot\frac{\partial \hat{f}_1}{\partial \vec{q}_1}$$

Proof of (*)

Define $F(\vec{q}) = \frac{1}{V}\int_{V(\vec{q})} d^d\vec{r}\, f(\vec{r})$

By definition of $\vec{\nabla}F$, for any infinitesimal $d\vec{\ell}$, $F(\vec{q}+d\vec{\ell}) - F(\vec{q}) = \vec{\nabla}F\cdot d\vec{\ell}$

But also $F(\vec{q}+d\vec{\ell}) - F(\vec{q}) = \frac{1}{V}\left[\underbrace{\int_{V(\vec{q}+d\vec{\ell})} f(\vec{r})\,d^d\vec{r}}_{\vec{r}=\vec{r}\,'+d\vec{\ell}\;\Rightarrow\;\int_{V(\vec{q})} f(\vec{r}\,'+d\vec{\ell})\,d^d\vec{r}\,'} - \int_{V(\vec{q})} f(\vec{r})\,d^d\vec{r}\right]$

$\vec{r}\,' = \vec{r}$

$$\Rightarrow F(\vec{q}+d\vec{\ell}) - F(\vec{q}) = \frac{1}{V}\int d^d\vec{r}\left[f(\vec{r}+d\vec{\ell}) - f(\vec{r})\right]$$

$$= \frac{1}{V}\int d^d\vec{r}\left[\vec{\nabla}f(\vec{r})\cdot d\vec{\ell}\right] = \left[\frac{1}{V}\int d^d\vec{r}\;\vec{\nabla}f(\vec{r})\right]\cdot d\vec{\ell}$$

$$\Rightarrow \vec{\nabla}\left(\frac{1}{V}\int_{V(\vec q)} f(\vec r\,')\, d^d \vec r\,'\right) = \frac{1}{V}\int_{V(\vec q)} \vec\nabla f(\vec r\,') \cdot d^d \vec r\,'$$

All in all

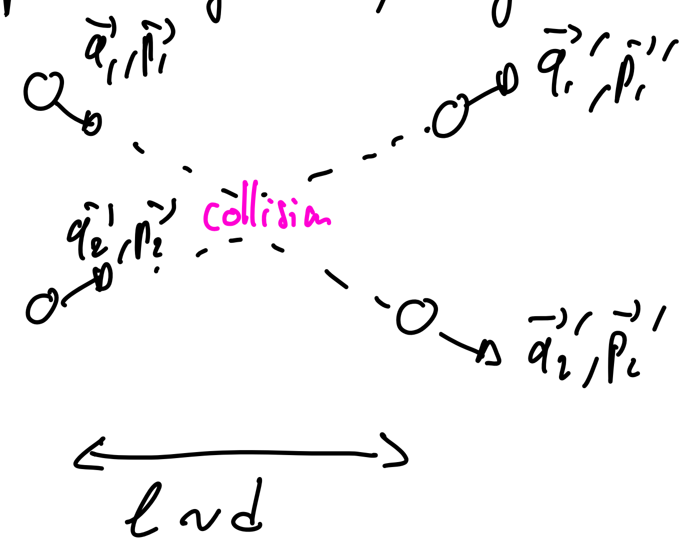
$$\partial_t \hat{f}_1\Big|_{FREE} = \{\hat{f}_1, H_1\}$$

B) The collisions

At this stage $\partial_t \hat f_1|_{coll}$ looks a mess & hard to analyse... let's coarse grain time!

2.2.2.2) Speeding up time

When two particles get close, they interact:



This happens on a scale of order the particle size & on a time scale of order τ_{col}.

Impact on $\hat{f}_1$

* While the impact of a collision on $\vec q_1, \vec p_1$ is typically large, such collisions are rare. On a time $t \sim \tau_{col}$, the proba-bility that such a collision takes place is very rare

$\Rightarrow$ the average probability to find a particle at $\vec{q}_1, \vec{p}_1$ thus barely evolves on $t \sim \tau_{col}$

* On a time $t \sim \tau_{MFP}$, there are many collisions
$\Rightarrow$ significant evolution of $\hat{f}_1$.

Scaling form: $\hat{f}_1(\vec{q}_1, \vec{p}_1, t) = g(\vec{q}_1, \vec{p}_1, \tilde{t} = \frac{t}{\tau_{MFP}})$

such that $\partial_{\tilde{t}}\, g(\vec{q}_1, \vec{p}_1, \tilde{t}) \sim \mathcal{O}(1)$

* To construct the dynamics of $\hat{f}_1$ on a timescale τ_{MFP}, we consider the evolution on $[t, t+\tau]$ with $\tau_{col} \ll \tau \ll \tau_{MFP}$.

Free transport on time τ

* $\displaystyle\int_t^{t+\tau} ds\, \{\hat{f}_1, H_1\} = \{\underbrace{\int_t^{t+\tau} ds\, \hat{f}_1}_{\sim \tau \hat{f}_1(\vec{q}_1, \vec{p}_1, t)}, H_1\}$ since H_1 is time independent since $\hat{f}_1$ barely evolves

Proof:

$$\hat{f}_1(\vec{q}_1, \vec{p}_1, s) = g\left(\vec{q}_1, \vec{p}_1, \frac{s}{\tau_{MFP}}\right)$$

$$\simeq g\left(\vec{q}_1, \vec{p}_1, \frac{t}{\tau_{MFP}}\right) + \frac{s-t}{\tau_{MFP}} \partial_{\tilde{t}}\, g + \frac{1}{2} \frac{(s-t)^2}{\tau_{MFP}^2} \partial_{\tilde{t}\tilde{t}}\, g \qquad (\Delta)$$

$$\int_t^{t+\tau} ds\, \hat{f}_1(\vec{q}_1, \vec{p}_1, s) = \underbrace{\tau g}_{\tau \hat{f}_1} + \underbrace{\frac{\tau^2}{2\tau_{MFP}} \partial_{\tilde{t}}\, g}_{\mathcal{O}\left(\tau \cdot \frac{\tau}{\tau_{MFP}}\right) \ll \tau} \Rightarrow \int_t^{t+\tau} ds\, \{\hat{f}_1, H_1\} \simeq \tau \{\hat{f}_1, H_1\}$$

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* $\int_t^{t+\tau} ds\, \partial_s \hat{f}_1(\vec{q}_1, \vec{p}_1, s) \simeq \tau \partial_t \hat{f}_1(\vec{q}_1, \vec{p}_1, t)$ since $\partial_t \hat{f}_1$ barely evolves

Proof: $\int_t^{t+\tau} ds\, \partial_s \hat{f}_1 \overset{(\Delta)}{=} \int_t^{t+\tau} ds \left[\frac{1}{\tau_{MFP}} \partial_{\tilde{t}} g + \frac{s-t}{\tau_{MFP}^2} \partial_{\tilde{t}\tilde{t}} g \right]$

$$= \underbrace{\frac{\tau}{\tau_{MFP}} \partial_{\tilde{t}} g\left(\vec{q}_1, \vec{p}_1, \frac{t}{\tau_{MFP}}\right)}_{\tau \partial_t \hat{f}_1\left(\vec{q}_1, \vec{p}_1, \frac{t}{\tau_{MFP}}\right)} + \underbrace{\frac{1}{2} \frac{\tau^2}{\tau_{MFP}^2} \partial_{\tilde{t}\tilde{t}} g}_{\mathcal{O}\left(\frac{\tau^2}{\tau_{MFP}^2}\right) \Rightarrow \text{negligible}}$$

$$\Rightarrow \int_t^{t+\tau} ds\, \partial_s \hat{f}_1 + \{\hat{f}_1, H_1\} \simeq \tau \left[\partial_t \hat{f}_1 + \{\hat{f}_1, H_1\} \right]\Big|_t$$

Collisions On a time $\tau_{col} \ll \tau \ll \tau_{MFP}$, there are very few collisions, which appear as rare & random encounters between particles $\Rightarrow$ build stochastic description